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KURTOSIS OF A RANDOM VECTOR SPECIAL TYPES OF DISTIBUTIONS

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ABSTRACT

In presented paper the authors attempt to generalize definition of kurtosis for the case of multidimensional and prove its essential properties. The generalized characteristic applied in the single-dimension case has the same properties as kurtosis, that is known in the literature on single-dimensional random variables. The basis of conducted considerations is the definition of *the power of a vector in space with the scalar product.*

Key words: Kurtosis, Moments of a random vector, Multidimensional distribution, Power of a vector.

1. Introduction

This paper is a continuation of previous studies by Tatar (1996,1999, 2000a, 2000b,2002, 2003), Osiewalski and Tatar (1999). Let us recoll:

Let there be any vector space $(R^n, R,+,\cdot)$ with the classical (Euclidean) scalar product form

$$\forall v = (v_1, v_2,..., v_n), w = (w_1, w_2,..., w_n) \in R^n : \langle v, w \rangle = \sum_{i=1}^{n} v_i w_i .$$

For any $v \in R^n$ and any number $k \in N_0 = N \cup \{0\}$ a following definition of *k*-th power of the vector *v* has been proposed.

Definition 1

$$v^o = 1 \in R$$

and

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$$v^k = \begin{cases} v^{k-1} \cdot v & , \quad for \;\; k - odd \\ \langle v^{k-1}, v \rangle & , \quad for \; k - even \end{cases}$$

From the above-mentioned definition immediately result two following important properties

$$\forall v \in R^n \;\; , \; k \in N_0 \;\; : \;\; k - even \quad \Rightarrow \quad v^k \in R$$
$$\forall v \in R^n \;\; , \; k \in N \;\; : \;\; k - odd \;\; \Rightarrow \;\; v^k \in R^n \; .$$

Definition 2

For any number $k \in N_0$ an ordinary moment of the order k of a random vector ξ we call the expected value of a random variables $g(\xi) = \xi^k$, it means $m_k = E(\xi^k)$, if $E(|\xi^k|) < +\infty$.

Particularly, for $n = 2$ and for any $k \in N$ we have (1)

$$m_k = \begin{cases} \sum_{i=0}^{k/2} m_{k-2i,2i} & , \; for \;\; k \text{ - even}, \\ \left(\sum_{i=0}^{(k-1)/2} \binom{\frac{k-1}{2}}{i} m_{k-2i,2i} \, , \; \sum_{i=0}^{(k-1)/2} \binom{\frac{k-1}{2}}{i} m_{k-2i,2i+1} \right) & , \; for \;\; k - odd \end{cases}$$

The symbols like $m_{k-2i,2i}$ used in the formula (1) means the classical mixed ordinary moments of the rank “ $k - 2i,2i$ “.

Similarly to the ordinary moments, we define central moments of multidimensional distribution. We only need to assume in the definition 2, that $g(\xi) = (\xi - E(\xi))^k$.

From the definition 1 and 2 results inter alia the following general conclusion:

$$\forall k \in N_0 : k \text{ - even} \Rightarrow m_k \in R$$

$$\forall k \in N : k \text{ - odd} \Rightarrow m_k \in R^n \quad .$$

In previously cited papers, on the basis of definition of the power of a vector and the ordinary moment of a random vector, important results for characterization of multidimensional probability distributions were obtained. Inter alia *Chebyshev’s Inequality* known for single-dimension case was generalized, multidimensional versions of the *law of large numbers* were proposed, there were defined and applied tools which help to analyze and measure the *asymmetry of* *multidimensional probability distributions*, the definition of *covariance and correlation coefficient* was generalized as well. We would like to emphasize that proposed and developed conception essentially and directly provides us with characteristics of analyzed random vector, not – properly constructed (single-dimensional) – functions of its terms.

To exercise our previous declaration, let's move to the proposition of measurement of concentration (which means flattening as well) probability distribution of a random vector.

2. Propositions

At the beginning let's recollect, that in the theory of single-dimensional random variables in capacity of concentration distribution measure around expected value it is assumed, inter alia, its fourth central moment. Usually – in order to obtain relative measure – the fourth order central moment is furthermore divided by square variance. The high values of indicator constructed in this way (*kurtosis*) indicate that there is the significant tendency to focus the distribution around its expected value and vice versa: indicators' low level shows its flattening.

The definition of random vector central moment let us to definite the kurtosis in a similar way for the multidimensional distribution.

Definition 3

The kurtosis of a random vector $X = (X_1, ..., X_n): \Omega \to R^n$ the following value will be called:

$$\mathrm{Kurt}X = \frac{E\left[(X - EX)^4\right]}{\left(D^2 X\right)^2} \quad . \tag{2}$$

In the proofs of theorems 1 and 2, which we'll form in the further part of this paper, we'll use the following lemma.

Lemma 1

Let $X = (X_1, ..., X_n): \Omega \to R^n$ be a random vector fulfilling conditions:

a) $X_i \sim N(m_i, \sigma_i^2)$, for any $i \in \{1,2,...,n\}$

b) for all $i, j \in \{1,2,...,n\}$: if $i \neq j$, than X_i and X_j are stochastically independent (symbolically: $X_i \perp X_j$).

Then we have the following equalities:

1. $E\left(\langle X, X\rangle^2\right) = \sum\limits_{\substack{i,j=1 \\ i \neq j}}^{n} \left(m_i^2 + \sigma_i^2\right)\left(m_j^2 + \sigma_j^2\right) + \sum\limits_{i=1}^{n}\left(3\sigma_i^4 + 6m_i^2\sigma_i^2 + m_i^4\right),$

2. $E\left(\langle X,X\rangle\langle X,EX\rangle\right)=\sum_{\substack{i,j=1\\ i\neq j}}^{n} m_j^2\left(m_i^2+\sigma_i^2\right)+\sum_{i=1}^{n}\left(3m_i^2\sigma_i^2+m_i^4\right),$

3. $E\left(\langle X,EX\rangle^2\right)=\sum_{\substack{i,j=1\\ i\neq j}}^{n} m_i^2 m_j^2+\sum_{i=1}^{n}\left(m_i^4+m_i^2\sigma_i^2\right),$

4. $\langle EX,EX\rangle E\left(\langle X,X\rangle\right)=\sum_{i,j=1}^{n}\left(m_i^2 m_j^2+m_i^2\sigma_j^2\right),$

5. $\langle EX,EX\rangle^2=\sum_{i,j=1}^{n} m_i^2 m_j^2 .$

Proof:

Let's recollect, that if a random variable $\xi:\Omega\to R$ has a normal distribution with a mean of *m* and a variance of σ^2, that is $\xi\sim N\left(m,\sigma^2\right)$, then

$$E\left(\xi^2\right)=m^2+\sigma^2\ ,\quad E\left(\xi^3\right)=3m\sigma^2+m^3$$

$$\text{and}\quad E\left(\xi^4\right)=3\sigma^4+6m^2\sigma^2+m^4 . \tag{3}$$

While using the properties of integral (alternatively sum) and the independence of random variables X_i^2 and X_j^2 for all $i,j\in\{1,2,...,n\}$, such that $i\neq j$ we get

$$E\left(\langle X,X\rangle^2\right)=E\left[\left(\sum_{i=1}^{n}X_i^2\right)\left(\sum_{j=1}^{n}X_j^2\right)\right]=\sum_{i,j=1}^{n}E\left(X_i^2X_j^2\right)= \tag{4}$$

$$=\sum_{\substack{i,j=1\\ i\neq j}}^{n}E\left(X_i^2\right)E\left(X_j^2\right)+\sum_{i=1}^{n}E\left(X_i^4\right).$$

Thus using (3) the equality (4) takes form:

$$E\left(\langle X,X\rangle^2\right)=\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(m_i^2+\sigma_i^2\right)\left(m_j^2+\sigma_j^2\right)+\sum_{i=1}^{n}\left(3\sigma_i^4+6m_i^2\sigma_i^2+m_i^4\right),$$

And that ends the proof for thesis 1.

Successively from the independence of random variables X_i^2 and X_j and form (3) of the ordinary moments of the second and third order a random variable with normal distribution, we get thesis 2.

In fact:

$$E\left(\langle X,X\rangle\langle X,EX\rangle\right)=E\left[\left(\sum_{i=1}^{n}X_i^2\right)\left(\sum_{j=1}^{n}X_jm_j\right)\right]=\sum_{i,j=1}^{n}E\left(X_i^2X_jm_j\right)=$$

$$=\sum_{\substack{i,j=1\\i\neq j}}^{n}m_jE\left(X_i^2\right)E\left(X_j\right)+\sum_{i=1}^{n}m_iE\left(X_i^3\right)=\sum_{\substack{i,j=1\\i\neq j}}^{n}m_j^2\left(m_i^2+\sigma_i^2\right)+\sum_{i=1}^{n}m_i\left(3m_i\sigma_i^2+m_i^3\right)=$$

$$=\sum_{\substack{i,j=1\\i\neq j}}^{n}m_j^2\left(m_i^2+\sigma_i^2\right)+\sum_{i=1}^{n}\left(3m_i^2\sigma_i^2+m_i^4\right).$$

In order to prove the thesis 3 we use the independence of random variables X_i and X_j as well. So there is:

$$E\left(\langle X,EX\rangle^2\right)=E\left[\left(\sum_{i=1}^{n}X_iEX_i\right)\left(\sum_{j=1}^{n}X_jEX_j\right)\right]=\sum_{i,j=1}^{n}E\left(X_iX_jE(X_i)E(X_j)\right)=$$

$$=\sum_{i,j=1}^{n}E(X_i)E(X_j)E(X_iX_j)=\sum_{\substack{i,j=1\\i\neq j}}^{n}m_i^2m_j^2+\sum_{i=1}^{n}\left(m_i^4+m_i^2\sigma_i^2\right).$$

Finally, for thesis 4 we only need to conduct the following reasoning:

$$\langle EX,EX\rangle E\left(\langle X,X\rangle\right)=\left(\sum_{i=1}^{n}m_i^2\right)E\left(\sum_{j=1}^{n}X_j^2\right)=\left(\sum_{i=1}^{n}m_i^2\right)\left(\sum_{j=1}^{n}E\left(X_j^2\right)\right)=$$

$$=\left(\sum_{i=1}^{n}m_i^2\right)\left(\sum_{j=1}^{n}\left(m_j^2+\sigma_j^2\right)\right)=\sum_{i,j=1}^{n}\left(m_i^2\left(m_j^2+\sigma_j^2\right)\right)=\sum_{i,j=1}^{n}\left(m_i^2m_j^2+m_i^2\sigma_j^2\right).$$

Thesis 5 is in this way obviously fulfilled.

Proof of the lemma has been finished.

At the moment, the theorem concerning the kurtosis of a random vector with normal marginal distributions, will be defined and proved.

Theorem 1

If $X=(X_1,...,X_n):\Omega\rightarrow R^n$ is a random vector fulfilling lemma 1 assumptions, then

$$KurtX=1+\frac{2\sum_{i=1}^{n}\sigma_i^4}{\sum_{i,j=1}^{n}\sigma_i^2\sigma_j^2}=1+\frac{2\sum_{i=1}^{n}\left(D^2X_i\right)^2}{\sum_{i,j=1}^{n}D^2X_iD^2X_j}. \qquad (5)$$

Proof:

Using the definition of even power of the vector, properties of scalar product and integral (sum), we get:

$$E\left[\left(X-EX\right)^{4}\right]=$$

$$E\left[\left\langle X-EX,X-EX\right\rangle^{2}\right]=E\left[\left(\left\langle X,X\right\rangle-2\left\langle X,EX\right\rangle+\left\langle EX,EX\right\rangle\right)^{2}\right]=$$

$$E\left[\left\langle X,X\right\rangle^{2}-4\left\langle X,X\right\rangle\left\langle X,EX\right\rangle+4\left\langle X,EX\right\rangle^{2}+2\left\langle X,X\right\rangle\left\langle EX,EX\right\rangle\right]+$$

$$-E\left[4\left\langle X,EX\right\rangle\cdot\left\langle EX,EX\right\rangle+\left\langle EX,EX\right\rangle^{2}\right].$$

Let's notice, that we also have equality:

$$E\left(\left\langle X,EX\right\rangle\left\langle EX,EX\right\rangle\right)=\left(\left\langle EX,EX\right\rangle\right)^{2} \tag{6}$$

Indeed:

$$E\left(\left\langle X,EX\right\rangle\left\langle EX,EX\right\rangle\right)=$$

$$=E\left[\left(\sum_{i=1}^{n}X_{i}EX_{i}\right)\left(\sum_{j=1}^{n}\left(EX_{j}\right)^{2}\right)\right]=E\left[\left(\sum_{i=1}^{n}X_{i}m_{i}\right)\left(\sum_{j=1}^{n}m_{j}^{2}\right)\right]=$$

$$=\left(\sum_{j=1}^{n}m_{j}^{2}\right)\left(\sum_{i=1}^{n}m_{i}EX_{i}\right)=\left(\sum_{j=1}^{n}m_{j}^{2}\right)\left(\sum_{i=1}^{n}m_{i}^{2}\right)=\left(\sum_{k=1}^{n}m_{k}^{2}\right)^{2}=\left(\left\langle EX,EX\right\rangle\right)^{2}.$$

Taking into consideration the equality (6), the fourth central moment can be shown in the following way

$$E\left[\left(X-EX\right)^{4}\right]=E\left(\left\langle X,X\right\rangle^{2}\right)-4E\left(\left\langle X,X\right\rangle\left\langle X,EX\right\rangle\right)+4E\left(\left\langle X,EX\right\rangle^{2}\right)+ \tag{7}$$

$$+2\left\langle EX,EX\right\rangle E\left(\left\langle X,X\right\rangle\right)-3\left\langle EX,EX\right\rangle^{2}.$$

Successively, from the lemma 1, equality (7) takes form:

$$E\left[\left(X-EX\right)^{4}\right]=\sum_{i,j=1}^{n}\sigma_{i}^{2}\sigma_{j}^{2}+2\sum_{i=1}^{n}\sigma_{i}^{4}\,. \tag{8}$$

Indeed:

$$E\left[(X-EX)^4\right]=\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(m_i^2+\sigma_i^2\right)\left(m_j^2+\sigma_j^2\right)+\sum_{i=1}^{n}\left(3\sigma_i^4+6m_i^2\sigma_i^2+m_i^4\right)+$$

$$-4\left(\sum_{\substack{i,j=1\\i\neq j}}^{n}m_j^2\left(m_i^2+\sigma_i^2\right)+\sum_{i=1}^{n}\left(3m_i^2\sigma_i^2+m_i^4\right)\right)+4\left(\sum_{\substack{i,j=1\\i\neq j}}^{n}m_i^2m_j^2+\sum_{i=1}^{n}\left(m_i^4+m_i^2\sigma_i^2\right)\right)+$$

$$+2\left(\sum_{i,j=1}^{n}\left(m_i^2m_j^2+m_i^2\sigma_j^2\right)\right)-3\sum_{i,j=1}^{n}m_i^2m_j^2=$$

$$=2\sum_{\substack{i,j=1\\i\neq j}}^{n}m_i^2\sigma_j^2+\sum_{\substack{i,j=1\\i\neq j}}^{n}m_i^2m_j^2+\sum_{\substack{i,j=1\\i\neq j}}^{n}\sigma_i^2\sigma_j^2+\sum_{i=1}^{n}\sigma_i^4+2\sum_{i=1}^{n}\sigma_i^4+2\sum_{i=1}^{n}m_i^2\sigma_i^2+$$

$$+4\sum_{i=1}^{n}m_i^2\sigma_i^2+\sum_{i=1}^{n}m_i^4-4\left(\sum_{\substack{i,j=1\\i\neq j}}^{n}m_j^2\left(m_i^2+\sigma_i^2\right)+\sum_{i=1}^{n}\left(3m_i^2\sigma_i^2+m_i^4\right)\right)+$$

$$+4\left(\sum_{\substack{i,j=1\\i\neq j}}^{n}m_i^2m_j^2+\sum_{i=1}^{n}\left(m_i^4+m_i^2\sigma_i^2\right)\right)+2\left(\sum_{i,j=1}^{n}\left(m_i^2m_j^2+m_i^2\sigma_j^2\right)\right)-3\sum_{i,j=1}^{n}m_i^2m_j^2=$$

$$=2\sum_{\substack{i,j=1\\i\neq j}}^{n}m_i^2\sigma_j^2+2\sum_{i=1}^{n}m_i^2\sigma_i^2+\sum_{\substack{i,j=1\\i\neq j}}^{n}m_i^2m_j^2+\sum_{i=1}^{n}m_i^4+\sum_{\substack{i,j=1\\i\neq j}}^{n}\sigma_i^2\sigma_j^2+\sum_{i=1}^{n}\sigma_i^4+$$

$$+2\sum_{i=1}^{n}\sigma_i^4+4\sum_{i=1}^{n}m_i^2\sigma_i^2-4\left(\sum_{i,j=1}^{n}m_i^2m_j^2+\sum_{i,j=1}^{n}m_i^2\sigma_j^2+2\sum_{i=1}^{n}m_i^2\sigma_i^2\right)+$$

$$+4\left(\sum_{i,j=1}^{n}m_i^2m_j^2+\sum_{i=1}^{n}m_i^2\sigma_i^2\right)+2\sum_{i,j=1}^{n}m_i^2\sigma_j^2+2\sum_{i,j=1}^{n}m_i^2m_j^2-3\sum_{i,j=1}^{n}m_i^2m_j^2=$$

$$=2\sum_{i,j=1}^{n}m_i^2\sigma_j^2+\sum_{i,j=1}^{n}m_i^2m_j^2+\sum_{i,j=1}^{n}\sigma_i^2\sigma_j^2+2\sum_{i=1}^{n}\sigma_i^4+4\sum_{i=1}^{n}m_i^2\sigma_i^2+$$

$$-4\sum_{i,j=1}^{n}m_j^2\sigma_i^2-8\sum_{i=1}^{n}m_i^2\sigma_i^2+4\sum_{i=1}^{n}m_i^2\sigma_i^2+2\sum_{i,j=1}^{n}m_i^2\sigma_j^2+2\sum_{i,j=1}^{n}m_i^2m_j^2-3\sum_{i,j=1}^{n}m_i^2m_j^2=$$

$$=4\sum_{i,j=1}^{n}m_i^2\sigma_j^2-4\sum_{i,j=1}^{n}m_j^2\sigma_i^2+4\sum_{i=1}^{n}m_i^2\sigma_i^2-8\sum_{i=1}^{n}m_i^2\sigma_i^2+4\sum_{i=1}^{n}m_i^2\sigma_i^2+$$

$$+\sum_{i,j=1}^{n} m_i^2 m_j^2 + 2\sum_{i,j=1}^{n} m_i^2 m_j^2 - 3\sum_{i,j=1}^{n} m_i^2 m_j^2 + \sum_{i,j=1}^{n} \sigma_i^2 \sigma_j^2 + 2\sum_{i=1}^{n} \sigma_i^4 = \sum_{i,j=1}^{n} \sigma_i^2 \sigma_j^2 + 2\sum_{i=1}^{n} \sigma_i^4 .$$

Furthermore, the following sequence of equalities is true:

$$\left(D^2 X\right)^2 = \left(\sum_{i=1}^{n} D^2 X_i\right)^2 = \left(\sum_{i=1}^{n} \sigma_i^2\right)^2 = \sum_{i,j=1}^{n} \sigma_i^2 \sigma_j^2 . \tag{9}$$

In the view of (8), (9) and from definition 3, kurtosis of a random vector, fulfilling lemma 1 assumptions, can be expressed as:

$$KurtX = 1 + \frac{2\sum_{i=1}^{n} \sigma_i^4}{\sum_{i,j=1}^{n} \sigma_i^2 \sigma_j^2} .$$

And that finishes the proof of theorem 1.

It's easy to notice that simple and immediate consequence of theorem 1 is the following conclusion.

Conclusion 1

If $X = (X_1,..., X_n): \Omega \to R^n$ is a random vector fulfilling assumptions of lemma 1, and if there is also condition $\sigma_1 = \sigma_2 = ... = \sigma_n = \sigma$, then

$$KurtX = 1 + \frac{2}{n} .$$

From the conclusion above also comes out – quite obvious, but it let us to call the submitted proposition as "generalization" – a remark that kurtosis of a single-dimensional random variable, normally distributed equals 3.

In order to realize the purpose formulated at the beginning of this paper, in the next theorem we'll get the form of kurtosis of a random vector, which marginal variables have a Student's t-distributions.

Theorem 2

Let $X = (X_1,..., X_n): \Omega \to R^n$ be a random vector fulfilling following conditions:

a) $X_i \sim t_{\nu_i}$ (that is X_i has a Student's t-distribution with ν_i degrees of freedom) where $\nu_i > 4$, for each $i \in \{1,2,...,n\}$,

b) for all $i, j \in \{1,2,...,n\}$: if $i \neq j$, then X_i and X_j are stochastically independent (symbolically: $X_i \perp X_j$).

Then

$$KurtX = 1 + \frac{\sum_{i=1}^{n}\left(2+\frac{6}{\nu_i - 4}\right)\left(\frac{\nu_i}{\nu_i - 2}\right)^2}{\sum_{i,j=1}^{n}\left(\frac{\nu_i}{\nu_i - 2}\right)\left(\frac{\nu_j}{\nu_j - 2}\right)} \quad . \tag{10}$$

Proof:

We'll conduct the reasoning similar to the one in the proof for theorem 1. But first, let's recollect, that if random variable $\xi : \Omega \to R$ has a Student's t-distribution with ν degrees of freedom (else: $\xi \sim t_\nu$), where $\nu > 4$, then:

$$E(\xi) = 0 \ , \quad E(\xi^2) = D^2\xi = \frac{\nu}{\nu - 2} \ , \quad E(\xi^3) = 0 \ ,$$

$$E(\xi^4) = \left(3 + \frac{6}{\nu - 4}\right)\left(\frac{\nu}{\nu - 2}\right)^2$$

and $Kurt\xi = 3 + \frac{6}{\nu - 4}$, $Excess\ \xi = \frac{6}{\nu - 4}$.

We also need to notice, that for vector X, which was discussed in the assumption of theorem 2, we have the equality $EX = (EX_1, EX_2, ..., EX_n) = (0,0,...,0)$. So here emerges conclusion that the thesis of lemma is equivalent for lemma 1 (with the difference that now we assume that the marginal variables of a random vector X have a Student's t-distribution) takes form:

1. $E\left(\langle X, X\rangle^2\right) = \sum_{\substack{i,j=1 \\ i\neq j}}^{n}\left(\frac{\nu_i}{\nu_i - 2}\right)\left(\frac{\nu_j}{\nu_j - 2}\right) + \sum_{i=1}^{n}\left(3 + \frac{6}{\nu_i - 4}\right)\left(\frac{\nu_i}{\nu_i - 2}\right)^2$,
2. $E\left(\langle X, X\rangle \cdot \langle X, EX\rangle\right) = 0$,
3. $E\left(\langle X, EX\rangle^2\right) = 0$,
4. $\langle EX, EX\rangle \cdot E\left(\langle X, X\rangle\right) = 0$,
5. $\langle EX, EX\rangle^2 = 0$.

Then:

$$E\left[\left(X-EX\right)^4\right]=E\left(\langle X,X\rangle^2\right)= \tag{11}$$

$$=\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(\frac{\nu_i}{\nu_i-2}\right)\left(\frac{\nu_j}{\nu_j-2}\right)+\sum_{i=1}^{n}\left(3+\frac{6}{\nu_i-4}\right)\left(\frac{\nu_i}{\nu_i-2}\right)^2 .$$

and

$$\left(D^2X\right)^2=\left(\sum_{i=1}^{n}\frac{\nu_i}{\nu_i-2}\right)^2=\sum_{i,j=1}^{n}\left(\frac{\nu_i}{\nu_i-2}\right)\left(\frac{\nu_j}{\nu_j-2}\right). \tag{12}$$

So from (11) and (12) and from definition 3 we get:

$$KurtX=\frac{\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(\frac{\nu_i}{\nu_i-2}\right)\left(\frac{\nu_j}{\nu_j-2}\right)+\sum_{i=1}^{n}\left(3+\frac{6}{\nu_i-4}\right)\left(\frac{\nu_i}{\nu_i-2}\right)^2}{\sum_{i,j=1}^{n}\left(\frac{\nu_i}{\nu_i-2}\right)\left(\frac{\nu_j}{\nu_j-2}\right)}=$$

$$=\frac{\sum_{i,j=1}^{n}\left(\frac{\nu_i}{\nu_i-2}\right)\left(\frac{\nu_j}{\nu_j-2}\right)-\sum_{i=1}^{n}\left(\frac{\nu_i}{\nu_i-2}\right)^2+\sum_{i=1}^{n}\left(3+\frac{6}{\nu_i-4}\right)\left(\frac{\nu_i}{\nu_i-2}\right)^2}{\sum_{i,j=1}^{n}\left(\frac{\nu_i}{\nu_i-2}\right)\left(\frac{\nu_j}{\nu_j-2}\right)}=$$

$$=\frac{\sum_{i,j=1}^{n}\left(\frac{\nu_i}{\nu_i-2}\right)\left(\frac{\nu_j}{\nu_j-2}\right)+\sum_{i=1}^{n}\left(2+\frac{6}{\nu_i-4}\right)\left(\frac{\nu_i}{\nu_i-2}\right)^2}{\sum_{i,j=1}^{n}\left(\frac{\nu_i}{\nu_i-2}\right)\left(\frac{\nu_j}{\nu_j-2}\right)}=$$

$$=1+\frac{\sum_{i=1}^{n}\left(2+\frac{6}{\nu_i-4}\right)\left(\frac{\nu_i}{\nu_i-2}\right)^2}{\sum_{i,j=1}^{n}\left(\frac{\nu_i}{\nu_i-2}\right)\left(\frac{\nu_j}{\nu_j-2}\right)},$$

and that is the required thesis (10).

Also from theorem 2 results the following conclusion:

Conclusion 2

If $X = (X_1,...,X_n): \Omega \to R^n$ is a random vector fulfilling assumptions of theorem 2 and if we have condition $\nu_1 = \nu_2 = ... = \nu_n = \nu > 4$ as well, then

$$KurtX = 1 + \frac{2 + \frac{6}{\nu - 4}}{n} .$$

Proof:

We only need to notice that in the view of equal number of degrees of freedom of all marginal variables the following equalities - from theorem 2 – are true:

$$KurtX = 1 + \frac{n\left(2 + \frac{6}{\nu - 4}\right)\left(\frac{\nu}{\nu - 2}\right)^2}{\left(n\frac{\nu}{\nu - 2}\right)^2} = 1 + \frac{2 + \frac{6}{\nu - 4}}{n} .$$

Obviously for n=1 the kurtosis of a random variable with a Student's t-distribution depends only on a number of its degrees of freedom and equals

$$KurtX = 3 + \frac{6}{\nu - 4} .$$

Let’s notice that from conclusion 1 and 2 we obviously get:

Conclusion 3

If $X = (X_1,...,X_n): \Omega \to R^n$ is a random vector fulfilling assumptions of lemma 1, but $\sigma_1 = \sigma_2 = ... = \sigma_n = \sigma$, and if $Y = (Y_1,...,Y_n): \Omega \to R^n$ is a random vector fulfilling assumptions of theorem 2 and we have condition $\nu_1 = \nu_2 = ... = \nu_n = \nu > 4$, then $KurtY > KurtX$.

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